ON THE INFLUENCE OF KEYWAYS ON THE STRESS DISTRIBUTION IN CYLINDRICAL SHAFTS*

BY

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1. When a keyway is cut in a cylindrical shaft, subject to torsion only, and of circular cross section, the maximum stress in the keyed part will considerably exceed that in the full part. To form an estimate of this increase, Filon† has investigated the torsion in a shaft the cross section of which is composed of confocal ellipses and hyperbolas. His results are expressed in infinite series of trigonometric and hyperbolic functions, and their numerical computation is necessarily somewhat laborious. Moreover, his numerical results (l. c., § 22–23) extend only up to a ratio of minor and major axes in the ellipse equal to $\tanh(\pi/2) = 0.917$, and this section differs too much from a circle to allow a sufficiently close numerical estimate of the stress ratio in the latter case.

In the present paper, a cross section will be considered which frequently occurs in practice: the shaft section is taken as a circle, and the keyway is formed by a circular arc intersecting the former orthogonally.

We shall first determine the exact formulas for the stress distribution, and then derive from these an approximate formula, adapted to practical use, in which the ratio of the maximum stresses in the keyed and full parts of the shaft is represented by the expression

$$\frac{2 - \frac{4}{\pi}\frac{b}{a} + \left(2 - \frac{4}{\pi}\right)\frac{b^2}{a^2}}{1 - 4\frac{b^2}{a^2}},$$

where a and b are the radii of shaft and keyway respectively. This expression is exact for $b = 0$ (and thus shows that a flaw or crack in the surface of a circular shaft has the effect of doubling the maximum stress) but is somewhat too great for $0 < b/a \leqq \frac{1}{4}$, this range of the ratio b/a being sufficient for all cases occurring in common practice.

* Presented to the Society, Feb. 26, 1916.

† L. N. G. Filon, *On the resistance to torsion of certain forms of shafting, with special reference to the effect of keyways*, Philosophical Transactions of the Royal Society, London, ser. A, vol. 193 (1900), pp. 309–352.

If a linear formula for the ratio of maximum stresses in the keyed and full parts of the shaft is desired, the above expression may be replaced by

$$2 - \frac{1}{5}\frac{b}{a},$$

which is greater than the preceding one for $0 < a/b \leqq \frac{1}{4}$.

2. With the notations of the figure, it is seen at once that

$$\tan \alpha = \frac{b}{a}, \tag{1}$$

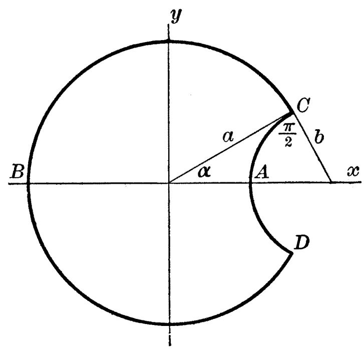

FIG. 1.

and that the distance between centers of shaft and keyway is $a \sec \alpha$.

The first step toward the determination of the torsional stresses consists in the construction of a harmonic function $\Psi(x, y)$* taking the value $\frac{1}{2}(x^2 + y^2)$ on the boundary of our cross section. To this purpose, write $z = x + iy$, $w = u + iv$, and map the area ABCD conformally on the first quadrant in the w-plane in such a manner that C corresponds to $w = 0$, D to $w = \infty$, the circular arc CBD to the u-axis, and the arc CAD to the v-axis. It is seen immediately that

$$w = \frac{z - ae^{\alpha i}}{e^{\alpha i} z - a}, \qquad z = a\frac{w - e^{\alpha i}}{e^{\alpha i} w - 1}, \qquad \frac{dw}{dz} = \frac{1}{a}\frac{(e^{\alpha i} w - 1)^2}{e^{2\alpha i} - 1}, \tag{2}$$

and the segment BA of the real axis corresponds to the first quadrant of the circle $u^2 + v^2 = 1$. From (2) we obtain

$$\tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}|z|^2 = \frac{a^2}{2}\frac{1 - 2\cos\alpha\cdot u - 2\sin\alpha\cdot v + u^2 + v^2}{1 - 2\cos\alpha\cdot u + 2\sin\alpha\cdot v + u^2 + v^2}, \tag{3}$$

* The notations are those of Love's *Theory of elasticity*, 2d edition, chapter XIV, except that his ξ, η, X_z, Y_z are replaced by u, v, X, Y.

so that the boundary conditions for Ψ become

$$\Psi = \tfrac{1}{2}|z|^2 = \begin{cases} \tfrac{1}{2}a^2 \text{ for } v = 0 \\ \tfrac{1}{2}a^2\left(1 - \dfrac{4\sin\alpha\cdot v}{1 + 2\sin\alpha\cdot v + v^2}\right) \text{ for } u = 0. \end{cases} \tag{4}$$

Now form Green's function $G(u', v'; u, v)$ in the first quadrant of the w-plane (u', v' are the coördinates of the variable point and u, v those of the pole). By reflecting the pole successively into the positive v-, the negative u-, and the negative v-axis, we obtain at once

$$G(u', v'; u, v) = \tfrac{1}{2}\log\frac{[(u'-u)^2 + (v'+v)^2][(u'+u)^2 + (v'-v)^2]}{[(u'-u)^2 + (v'-v)^2][(u'+u)^2 + (v'+v)^2]}.$$

The formula

$$\Psi = \frac{1}{2\pi}\int \Psi_s \frac{\partial G}{\partial n}\, ds$$

now gives

$$\frac{2\pi}{a^2}\Psi = \pi - \int_0^\infty \frac{4\sin\alpha\cdot v'}{1 + 2\sin\alpha\cdot v' + v'^2}\left[\frac{u}{u^2 + (v'-v)^2} - \frac{u}{u^2 + (v'+v)^2}\right] dv'. \tag{5}$$

whence, introducing the function Φ conjugate to Ψ,

$$\begin{aligned} \Phi + i\Psi = {} & \frac{a^2 i}{2} \\ & + \frac{2a^2\tan\alpha}{\pi}\,\frac{\pi - 2\alpha - \pi\cos\alpha\cdot w - (\pi - 2\alpha)\cos 2\alpha\cdot w^2 + \pi\cos\alpha\cdot w^3}{1 - 2\cos 2\alpha\cdot w^2 + w^4} \\ & - \frac{8a^2\sin^2\alpha}{\pi}\,\frac{w^2\log w}{1 - 2\cos 2\alpha\cdot w^2 + w^4} \end{aligned} \tag{6}$$

for $0 < \alpha < \pi/2$, where $\log w$ is the principal value of the logarithm.

For $v = 0$, $w = u$ and we obtain from (6)

$$\begin{aligned} \Phi = {} & \frac{2a^2\tan\alpha}{\pi}\left[\alpha\frac{\cos\alpha\cdot u - 1}{1 - 2\cos\alpha\cdot u + u^2} + (\pi - \alpha)\frac{\cos\alpha\cdot u + 1}{1 + 2\cos\alpha\cdot u + u^2}\right] \\ & - \frac{8a^2\sin^2\alpha}{\pi}\,\frac{u^2\log u}{1 - 2\cos 2\alpha\cdot u^2 + u^4}, \\ \Psi = {} & \frac{a^2}{2}; \end{aligned} \tag{7}$$

while for $u = 0$, $w = iv$, $\log w = \log v + \pi i/2$ and

$$\begin{aligned} \Phi &= \frac{2a^2\tan\alpha}{\pi}\,\frac{(\pi - 2\alpha)(1 + \cos 2\alpha\cdot v^2)}{1 + 2\cos 2\alpha\cdot v^2 + v^4} + \frac{8a^2\sin^2\alpha}{\pi}\,\frac{v^2\log v}{1 + 2\cos 2\alpha\cdot v^2 + v^4}, \\ \Psi &= \frac{a^2}{2} - \frac{2a^2\sin\alpha\cdot v}{1 + 2\sin\alpha\cdot v + v^2}; \end{aligned} \tag{8}$$

the identity of these boundary values of Ψ with those given in (4) serves as a check on the calculation.

3. We now proceed to determine the maximum stress in the cross section. The stress components X and Y are given by

$$X = \mu\tau\left(\frac{\partial\Phi}{\partial x} - y\right) = \mu\tau\left(\frac{\partial\Psi}{\partial y} - y\right),$$

$$Y = \mu\tau\left(\frac{\partial\Phi}{\partial y} + x\right) = -\mu\tau\left(\frac{\partial\Psi}{\partial x} - x\right),$$

and the maximum stress is the maximum of $\sqrt{X^2 + Y^2}$, which by an elementary property of harmonic functions must occur on the boundary.* For $u = 0$, we have

$$X = \mu\tau\frac{\partial}{\partial y}[\Psi - \tfrac{1}{2}(x^2 + y^2)]$$

$$= \mu\tau\left\{\frac{\partial}{\partial u}[\Psi - \tfrac{1}{2}(x^2 + y^2)]\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}[\Psi - \tfrac{1}{2}(x^2 + y^2)]\frac{\partial v}{\partial y}\right\}_{u=0}$$

and since $\Psi - \frac{1}{2}(x^2 + y^2) = 0$ for $u = 0$, the partial derivative of this expression in respect to v vanishes, so that

$$X = \mu\tau\frac{\partial u}{\partial y}\frac{\partial}{\partial u}[\Psi - \tfrac{1}{2}(x^2 + y^2)]_{u=0}$$

$$= -\mu\tau\frac{\partial u}{\partial y}\left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0}.$$

Similarly

$$Y = -\mu\tau\frac{\partial}{\partial x}[\Psi - \tfrac{1}{2}(x^2 + y^2)]_{u=0}$$

$$= \mu\tau\frac{\partial u}{\partial x}\left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0},$$

whence

$$\sqrt{X^2 + Y^2} = \mu\tau\left|\frac{dz}{dw}\right| \cdot \left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0}.$$

From (8), (3), and the last of (2), we now obtain

$$f(v) = \frac{\pi\cos\alpha}{\mu\tau a}\sqrt{X^2 + Y^2}$$

$$(9)\qquad = \frac{2(\pi - 2\alpha)v(\cos 2\alpha + 2v^2 + \cos 2\alpha\cdot v^4)}{(1 + 2\sin\alpha\cdot v + v^2)(1 - 2\sin\alpha\cdot v + v^2)^2}$$
$$- \frac{2\sin 2\alpha\,[v(1 + 2\cos 2\alpha\cdot v^2 + v^4) + 2v(1 - v^4)\log v]}{(1 + 2\sin\alpha\cdot v + v^2)(1 - 2\sin\alpha\cdot v + v^2)^2}$$
$$+ \frac{\pi(1 + \cos 2\alpha)v}{1 + 2\sin\alpha\cdot v + v^2}.$$

* See Love, *Theory of elasticity*, 2d ed., § 219.

The stress distribution being evidently symmetrical about the x-axis, the stresses on the arc AC are, in particular, equal to those on the arc AD, and this circumstance finds its expression in the fact that (8) is invariant when v is replaced by its reciprocal.

From $f(v)=f(1/v)$ it follows that $f'(v) = -f'(1/v)/v^2$, whence $f'(1)=0$, so that $f(v)$ becomes a maximum or minimum at $v=1$. To find any other maxima or minima, it is sufficient to consider the interval $0 \leqq v \leqq 1$; writing

$$v+\frac{1}{v}=2t, \qquad \frac{dt}{dv}=\frac{1}{2}\left(1-\frac{1}{v^2}\right) \tag{10}$$

so that $t \geqq 1$, (9) takes the form

$$2f(v) = \frac{Et^2-2Ft+G+2\sin 2\alpha\cdot t\left(v-\frac{1}{v}\right)\log v}{(t+\sin\alpha)(t-\sin\alpha)^2},$$

where

$$E = \pi+(3\pi-4\alpha)\cos 2\alpha-2\sin 2\alpha,$$

$$F = \pi\sin\alpha(1+\cos 2\alpha),$$

$$G = \sin^2\alpha(3\pi-4\alpha+\pi\cos 2\alpha+2\sin 2\alpha).$$

The equation $f'(v)=0$ now becomes

$$\begin{aligned}&\{2(t^2-\sin^2\alpha)[(E+2\sin 2\alpha)t-F]\\&\quad-(3t+\sin\alpha)[Et^2-2Ft+G]\}\left(1-\frac{1}{v^2}\right)\\&\quad+2\sin 2\alpha\cdot\log v\left\{(t^2-\sin^2\alpha)\left[\left(v-\frac{1}{v}\right)\left(1-\frac{1}{v^2}\right)\right.\right.\\&\quad\left.\left.+\left(v+\frac{1}{v}\right)\left(1+\frac{1}{v^2}\right)\right]-(3t+\sin\alpha)t\left(v-\frac{1}{v}\right)\left(1-\frac{1}{v^2}\right)\right\}=0,\end{aligned}$$

or reducing by (10) and dividing by $1-1/v^2$, thus removing the known root $v=1$,

$$\begin{aligned}\chi(v) = {}& 2(t^2-\sin^2\alpha)[(E+2\sin 2\alpha)t-F]\\&-(3t+\sin\alpha)[Et^2-2Ft+G]\\&+8\sin 2\alpha[(t^2-\sin^2\alpha)(2t^2-1)\\&-(3t+\sin\alpha)(t^3-t)]\frac{\log v}{v-\frac{1}{v}}=0.\end{aligned} \tag{11}$$

We may prove that this equation has no root between 0 and 1 as follows.

Since any root of $\chi(v) = 0$ in the interval $0 < v < 1$ varies continuously with α, three cases are to be considered when we let α decrease steadily from the given value to zero: (a) there exists some value $\alpha_0 \geqq 0$ of α such that the root in question approaches zero as $\alpha \to \alpha_0$; (b) the root in question approaches the value $v = 1$ for some value of α, or (c) the root will remain within the interval $0 < v < 1$ as α decreases toward zero. It is readily shown that all three cases are impossible. In case (a) we find from (11)

$$\limsup_{\substack{v \to 0 \\ \alpha \to \alpha_0}} \frac{\chi(v)}{t^3} = [2(E + 2 \sin 2\alpha) - 3E]_{\alpha=\alpha_0} - 4 \liminf_{\substack{v \to 0 \\ \alpha \to \alpha_0}} \sin 2\alpha \cdot \frac{\frac{1}{v} + v}{\frac{1}{v} - v} \log \frac{1}{v},$$

and since

$$\frac{\frac{1}{v} + v}{\frac{1}{v} - v} \log \frac{1}{v} > \log \frac{1}{v} > 1 \quad \text{for} \quad v < e^{-1},$$

we have

$$\liminf_{\substack{v \to 0 \\ \alpha \to \alpha_0}} \sin 2\alpha \cdot \frac{\frac{1}{v} + v}{\frac{1}{v} - v} \log \frac{1}{v} \geqq \sin 2\alpha_0,$$

so that finally

$$\limsup_{\substack{v \to 0 \\ \alpha \to \alpha_0}} \frac{\chi(v)}{t^3} \leqq - E_{\alpha=\alpha_0} < 0 \quad \text{for} \quad 0 \leqq \alpha_0 < \frac{\pi}{2}.$$

In case (b), we obtain at once

$$\chi(1) = -2 \cos^2 \alpha [(2 + 4 \sin \alpha)(\pi - \alpha) - 4 \sin 2\alpha + \sin^2 \alpha (9 - 2 \cos \alpha - \pi \sin \alpha - 8\alpha)],$$

and since $9 - 2 \cos \alpha - \pi \sin \alpha - 8\alpha$ decreases as α increases from 0 to $\pi/2$, we have

$$\chi(1) < -2 \cos^2 \alpha [(2 + 4 \sin \alpha)(\pi - \alpha) - 4 \sin \alpha - (5\pi - 9) \sin^2 \alpha].$$

In the interval $0 \leqq \alpha \leqq \pi/6$, we find

$$(2 + 4 \sin \alpha)(\pi - \alpha) - 4 \sin \alpha - (5\pi - 9) \sin^2 \alpha > 2(\pi - \pi/6) - 4 \cdot \tfrac{1}{2} - (5\pi - 9) \cdot \tfrac{1}{4} > 0,$$

and a similar argument applies to each of the intervals $\pi/6 \leqq \alpha \leqq \pi/3$, $\pi/3 \leqq \alpha \leqq \arcsin 4/5$, $\arcsin 4/5 \leqq \alpha \leqq \pi/2$, so that finally

$$\chi(1) < 0, \qquad 0 \leqq \alpha < \pi/2.$$

Hence $f''(1) < 0$, so that $v = 1$ gives a maximum of $\sqrt{X^2 + Y^2}$. In the third case, it is seen that for $0 < v < 1$

$$\lim_{\alpha \to 0} \chi(v) = -4\pi t^3 < 0,$$

so that none of the three cases yields a root of (11). For $v = 0$, (9) shows that $\sqrt{X^2 + Y^2} = 0$, and since $\chi(v) = 0$ has no root in the interval $0 < v < 1$, the maximum of $\sqrt{X^2 + Y^2}$ occurs at $v = 1$ for $0 \leqq \alpha < \pi/2$, and its value is found from (9) to be

$$\text{(12)} \quad \text{Max.}\ \sqrt{X^2 + Y^2} = \frac{\mu \tau a}{\pi} \frac{2(\pi - \alpha) - 2\pi \sin\alpha + \pi \sin^2\alpha - 2 \sin\alpha \cos\alpha}{\cos\alpha\,(1 - \sin\alpha)}.$$

Referring to the figure, the maximum stress on the arc CAD is therefore seen to occur at A, while the stress is zero at C and D. On the remaining part of the boundary, $v = 0$, and here we find, in the same way as for $u = 0$,

$$\sqrt{X^2 + Y^2} = \mu\tau \left|\frac{dw}{dz}\right| \left[\frac{\partial \Phi}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}(x^2 + y^2)\right]_{v=0}.$$

Treating this expression in exactly the same way as the preceding one, we find a unique maximum value of the stress at $u = 1$ (corresponding to the point B), which however is smaller than (12), the latter therefore giving the absolute maximum.

4. To determine the constant τ, we have the following relation, where M is the moment of the external forces about the axis of the shaft,

$$M = 2\mu\tau \iint [\Psi - \tfrac{1}{2}(x^2 + y^2)]\, dx\, dy.$$

Denoting by $I(u, v)$ the integral in (5), this becomes

$$M = \mu\tau a^2 \iint dx\, dy - \mu\tau \iint (x^2 + y^2)\, dx\, dy - \frac{\mu\tau a^2}{\pi} \iint I(u, v)\, dx\, dy,$$

or changing the integration variables in the last integral and remembering that

$$\frac{\partial(x, y)}{\partial(u, v)} = \left|\frac{dz}{dw}\right|^2$$

which may be found from (2), we obtain

$$\text{(13)} \quad \begin{aligned} M = {} & \mu\tau a^2 \iint dx\, dy - \mu\tau \iint (x^2 + y^2)\, dx\, dy \\ & - \frac{4\mu\tau a^4 \sin^2\alpha}{\pi} \int_0^\infty \int_0^\infty \frac{I(u, v)\, du\, dv}{(1 - 2\cos\alpha \cdot u + 2 \sin\alpha \cdot v + u^2 + v^2)^2}. \end{aligned}$$

The first integral in (13) equals the area of the cross section, and is most readily found by drawing CD and evaluating the area as the difference of the two circular segments CDBC and CDAC, of radii a and $a \tan \alpha$ and center angles $2\pi - 2\alpha$ and $\pi - 2\alpha$ respectively. Thus we find

$$\begin{aligned}\iint dx\,dy &= (\pi - \alpha + \sin\alpha\cos\alpha)\,a^2 - \left(\frac{\pi}{2} - \alpha - \cos\alpha\sin\alpha\right)a^2\tan^2\alpha \\ &= a^2\left(\pi - \alpha + \tan\alpha - \frac{\pi}{2}\tan^2\alpha + \alpha\tan^2\alpha\right).\end{aligned} \tag{14}$$

The second integral in (13) is the polar moment of inertia; introducing polar coördinates r, θ, the equation of the circular arc of radius b is

$$r = a \sec\alpha\,(\cos\theta - \sqrt{\cos^2\theta - \cos^2\alpha}),$$

and we have

$$\begin{aligned}\iint (x^2 + y^2)\,dx\,dy &= 2\int_\alpha^\pi d\theta \int_0^a r^3\,dr + 2\int_0^\alpha d\theta \int_0^r r^3\,dr \\ &= \tfrac{1}{2}a^4(\pi - \alpha) + \tfrac{1}{2}a^4\sec^4\alpha \int_0^\alpha (\cos\theta - \sqrt{\cos^2\theta - \cos^2\alpha})^4\,d\theta \\ &= \tfrac{1}{2}a^4(\pi - \alpha) + \tfrac{1}{2}a^4\sec^4\alpha \int_0^\alpha (8\cos^4\theta - 8\cos^2\cos\alpha^2\theta \\ &\quad + \cos^4\alpha)\,d\theta + 2a^4\sec^4\alpha \int_0^\alpha \cos\theta\,(2\sin^2\theta - 1 \\ &\quad - \sin^2\alpha)\sqrt{\sin^2\alpha - \sin^2\theta}\,d\theta.\end{aligned}$$

Making $\sin\theta = \sin\alpha\sin\phi$ in the last integral, the integrations may be performed, and expressing all trigonometric functions in $\tan\alpha$, we obtain

$$\begin{aligned}\iint (x^2 + y^2)\,dx\,dy = \frac{a^4}{2}\Big[\pi - \alpha + \tan\alpha - \pi\tan^2\alpha + 3\tan^3\alpha \\ + \alpha\tan^2\alpha - \frac{3\pi}{2}\tan^4\alpha + 3\alpha\tan^4\alpha\Big].\end{aligned} \tag{15}$$

In the last integral in (13), $I(u, v)$ may be obtained explicitly from (6), but the further integration in respect to u and v is not possible in finite terms.

5. We shall now derive approximate formulas, adapted to numerical calculation, from the purely theoretical results of the preceding paragraphs, and begin with the expression for the maximum stress. Equation (12) may be written

$$\begin{aligned}\text{Max. } \sqrt{X^2 + Y^2} &= 2\mu\tau a - \frac{4\mu\tau a}{\pi}\tan\alpha + \frac{\mu\tau a}{\pi} \\ &\times \frac{[2\pi - (2\pi - 2)\sin\alpha](1 - \cos\alpha) - 2\alpha + 2\sin\alpha + (\pi - 4)\sin^2\alpha}{\cos\alpha\,(1 - \sin\alpha)};\end{aligned}$$

replacing α by $\sin\alpha$, the last expression is increased (except for $\alpha = 0$) and we find

$$\begin{aligned}\text{Max.}\ \sqrt{X^2+Y^2} \leqq 2\mu\tau a &- \frac{4\mu\tau a}{\pi}\tan\alpha \\ &+ \frac{\mu\tau a}{\pi}\tan^2\alpha\left[\frac{(2\pi-2)\cos\alpha}{1+\cos\alpha} + \frac{2\cos\alpha}{(1-\sin\alpha)(1+\cos\alpha)}\right. \\ &\left. + \frac{(\pi-4)\cos\alpha}{1-\sin\alpha}\right].\end{aligned}$$

Now each of the three terms in the bracket decreases as α increases (as is seen upon forming their derivatives), and the value of the bracket for $\alpha = 0$ being $2\pi - 4$, we obtain

$$\text{(16)}\qquad \text{Max.}\ \sqrt{X^2+Y^2} \leqq \mu\tau a\left[2 - \frac{4}{\pi}\tan\alpha + \left(2 - \frac{4}{\pi}\right)\tan^2\alpha\right].$$

Next, we proceed to find an upper bound for the third integral in (13). From (5) it follows that, the expression in brackets under the integral sign being evidently positive, we have for $0 < \alpha < \pi/2$

$$I(u, v) < \sin\alpha\int_0^\infty \frac{4v'}{1+v'^2}\left[\frac{u}{u^2+(v'-v)^2} - \frac{u}{u^2+(v'+v)^2}\right]dv',$$

and performing the integration,

$$I(u, v) < \frac{4\pi\sin\alpha\cdot v}{(1+u)^2+v^2}.$$

Hence

$$\begin{aligned}\int_0^\infty &\frac{I(u, v)\,dv}{(1-2\cos\alpha\cdot u+2\sin\alpha\cdot v+u^2+z^2)^2} \\ &< \int_0^\infty \frac{4\pi\sin\alpha\cdot v\,dv}{[(1+u)^2+v^2][1-2\cos\alpha\cdot u+u^2+v^2]^2} \\ &= \frac{\pi\sin\alpha}{1+\cos\alpha}\left[\frac{1}{u}\cdot\frac{1}{1-2\cos\alpha\cdot u+u^2}\right. \\ &\qquad\left. + \frac{1}{2(1+\cos\alpha)}\cdot\frac{1}{u^2}\log\frac{1-2\cos\alpha\cdot u+u^2}{(1+u)^2}\right]\end{aligned}$$

and furthermore

$$\begin{aligned}\int_0^\infty\int_0^\infty &\frac{I(u, v)\,du\,dv}{(1-2\cos\alpha\cdot u+2\sin\alpha\cdot v+u^2+v^2)} \\ &< \frac{\pi\sin\alpha}{1+\cos\alpha}\int_0^\infty\left[\frac{1}{u}\cdot\frac{1}{1-2\cos\alpha\cdot u+u^2} + \frac{1}{2(1+\cos\alpha)}\right. \\ &\qquad\left.\cdot\frac{1}{u^2}\log\frac{1-2\cos\alpha\cdot u+u^2}{(1+u)^2}\right]du.\end{aligned}$$

Integrating by parts so as to remove the logarithm, the last integral is readily evaluated, and we finally obtain

$$
(17) \qquad \int_0^\infty \int_0^\infty \frac{I(u, v)\, du\, dv}{(1 - 2\cos\alpha\cdot u + 2\sin\alpha\cdot v + u^2 + v^2)^2} \leqq \frac{\pi(\pi - \alpha - \sin\alpha)(1 - \cos\alpha)}{\sin^2\alpha}
$$

for $0 \leqq \alpha < \pi/2$, the equality sign holding in (16) and (17) for $\alpha = 0$ only. From (13), (14), (15), and (17) it is now seen that

$$
M \geqq \frac{\mu\tau a^4}{2}[\pi - \alpha + \tan\alpha + \alpha\tan^2\alpha - 3\tan^3\alpha + 3\pi\tan^4\alpha - 3\alpha\tan^4\alpha - 8(\pi - \alpha - \sin\alpha)(1 - \cos\alpha)].
$$

We now use the inequalities

$$
\tan\alpha - \tfrac{1}{3}\tan^3\alpha < \alpha < \tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha < \tan\alpha,
$$

$$
\sin\alpha = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}} > \tan\alpha - \tfrac{1}{2}\tan^3\alpha,
$$

$$
1 - \cos\alpha = 1 - \frac{1}{\sqrt{1 + \tan^2\alpha}} < \tfrac{1}{2}\tan^2\alpha,
$$

valid for $0 < \tan\alpha < 1$, and obtain

$$
\begin{aligned}
M \geqq \frac{\mu\tau a^4}{2}[\pi &- (\tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha) + \tan\alpha \\
&+ (\tan\alpha - \tfrac{1}{3}\tan^3\alpha)\tan^2\alpha - 3\tan^3\alpha + 3\pi\tan^4\alpha - 3\tan\alpha\cdot\tan^4\alpha \\
&- 8(\pi - (\tan\alpha - \tfrac{1}{3}\tan^3\alpha) - (\tan\alpha - \tfrac{1}{2}\tan^3\alpha))\cdot\tfrac{1}{2}\tan^2\alpha \\
= \frac{\mu\pi\tau a^4}{2}&\left\{1 - \tan^2\alpha\left[4 - \frac{19}{3\pi}\tan\alpha\left(1 + \frac{9}{19}\tan\alpha - \frac{103}{45}\tan^2\alpha\right)\right]\right\}.
\end{aligned}
$$

Introducing the assumption $0 \leqq \tan\alpha \leqq \frac{1}{4}$, whence $0 \leqq \alpha \leqq 14^\circ\, 02'\, 10''.48$, the innermost parenthesis is positive, so that finally

$$
(18) \qquad M \geqq \frac{\mu\pi\tau a^4}{2}(1 - 4\tan^2\alpha).
$$

From (16) and (18) it is seen that, replacing $\tan\alpha$ by b/a, for $0 \leqq b/a \leqq \frac{1}{4}$,

$$
\text{Max. } \sqrt{X^2 + Y^2} \leqq \frac{2M}{\pi a^3}\cdot\frac{2 - \dfrac{4}{\pi}\dfrac{b}{a} + \left(2 - \dfrac{4}{\pi}\right)\dfrac{b^2}{a^2}}{1 - 4\dfrac{b^2}{a^2}}.
$$

For the full part of the shaft (cross section a circle of radius a) we have the well known formulas

$$\text{Max. } \sqrt{X^2 + Y^2} = \mu\tau a = \frac{2M}{\pi a^3}, \qquad M = \frac{\mu\pi\tau a^4}{2},$$

and consequently *the ratio of maximum stresses in the keyed and full parts of the shaft is less than*

$$\frac{2 - \frac{4}{\pi}\frac{b}{a} + \left(2 - \frac{4}{\pi}\right)\frac{b^2}{a^2}}{1 - 4\frac{b^2}{a^2}}, \tag{19}$$

for $0 < b/a \leqq \frac{1}{4}$, but equal to this expression ($= 2$) for $b = 0$. For practical use, the values of (19) may be tabulated, or we may replace (19) by a linear expression $2 + \lambda(b/a)$, where λ is to be given the least value for which this expression is greater than or equal to (19) throughout the interval $0 \leqq b/a \leqq \frac{1}{4}$. This leads to the condition

$$\lambda \leqq \frac{\left(6 - \frac{4}{\pi}\right)\frac{b}{a} - \frac{4}{\pi}}{1 - 4\frac{b^2}{a^2}}$$

and the right hand member decreasing as b/a increases from 0 to $\frac{1}{4}$, we find

$$\lambda \geqq \frac{\left(6 - \frac{4}{\pi}\right)\cdot\frac{1}{4} - \frac{4}{\pi}}{1 - \frac{1}{4}} = -0.2122,$$

so that the ratio of maximum stresses never exceeds

$$2 - 0.2122\frac{b}{a} \tag{20}$$

in the interval $0 \leqq b/a \leqq \frac{1}{4}$.